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Inequivalent surface-knots with the same knot quandle

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Dedicated to Professor Takao Matumoto on the occasion of his 60th birthday

Abstract

We have a knot quandle and a fundamental class as invariants for a surface-knot. These invariants can be defined for a classical knot in a similar way, and it is known that the pair of them is a complete invariant for classical knots. In surface-knot theory the situation is different: There exist arbitrarily many inequivalent surface-knots of genus g with the same knot quandle, and there exist two inequivalent surface-knots of genus g with the same knot quandle and with the same fundamental class.

MSC: 57Q45; 57M25

Keywords: Surface-knot; Quandle; Knot quandle; Fundamental class

1. Introduction

We consider a *knot quandle* [19,22], $Q(F)$, and a *fundamental class* [5] (cf. [31]), $[F] \in H_3^Q(Q(F))$, as invariants of a surface-knot F, where a surface-knot means an oriented closed connected surface embedded locally flatly in $\mathbb{R}^4$ (or in the 4-sphere S^4). The fundamental class can be considered as a universal object for *quandle cocycle invariant*s (see Section 2.2). When the invariants are given, what we want to know might be the following:

- What kind of information can be extracted from them?
- How powerful are they?

For the first question, it is shown in [19,22] that the knot quandle of a surface-knot F can recover all information of the knot group $\pi_1(\mathbb{R}^4 \setminus F)$, for example. There are some relations of the knot quandle to the braid index [30], to the unknotting number [18] and to the sheet number [25]. There are also some relations of the fundamental class to the noninvertibility [3,1,17], to the triple point number [27,28,15,31], to the triple point canceling number [18], and to the ribbon concordance [7].

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For the second question, it is shown in [2] that the knot quandle can distinguish all elements of a class of twist-spun S^2-knots obtained from torus knots, for example. In this paper, we focus on the second question and compare a situation in surface-knot theory with that in classical knot theory.

1.1. The case of classical knots

Similarly to the case of the surface-knot, we have a knot quandle $Q(k)$ and a fundamental class $[k] \in H_2^{\mathrm{Q}}(Q(k))$ as invariants of a classical knot k (cf. [8]). For a classical knot k, let $-k$ denote the classical knot obtained from k by reversing the orientation, and k^* denote the mirror image of k. Then the following three facts are known.

- Fact (cf. the proof of [4, Theorem 9.1]):
  For a classical knot k, there exists a canonical quandle isomorphism $\phi : Q(k) \to Q(-k^*)$ such that the induced homomorphism $\phi_* : H_2^{\mathrm{Q}}(Q(k)) \to H_2^{\mathrm{Q}}(Q(-k^*))$ satisfies the condition $\phi_*[k] = -[-k^*]$.
- Theorem due to Joyce [19] and Matveev [22]:
  For classical knots k and k', if there exists a quandle isomorphism $\phi : Q(k) \to Q(k')$, then k is equivalent to k' or $-(k')^*$.
- Theorem due to Eisermann [8]:
  For classical knots k and k', if there exists a quandle isomorphism $\phi : Q(k) \to Q(k')$ such that the induced homomorphism ϕ_* satisfies the condition $\phi_*[k] = [k']$, then k is equivalent to k'.

Roughly speaking, Joyce–Matveev's theorem says that the knot quandle is an almost complete invariant for classical knots, and Eisermann's theorem says that the pair of the knot quandle and the fundamental class is a complete invariant for them.

Remark 1. Eisermann [8] also proved:

- For a trivial classical knot k, we have $H_2^{\mathrm{Q}}(Q(k)) \cong 0$.
- For a nontrivial classical knot k, we have $H_2^{\mathrm{Q}}(Q(k)) \cong \mathbb{Z}$ and the fundamental class $[k]$ is a generator.

1.2. Problems

For a surface-knot F, let $-F$ denote the surface-knot obtained from F by reversing the orientation, and F^* denote the mirror image of F. It is known that the assertion corresponding to the first fact in Section 1.1 also holds for a surface-knot F, that is, there exists a canonical quandle isomorphism $\phi : Q(F) \to Q(-F^*)$ such that the induced homomorphism $\phi_* : H_3^{\mathrm{Q}}(Q(F)) \to H_3^{\mathrm{Q}}(Q(-F^*))$ satisfies the condition $\phi_*[F] = -[-F^*]$ (cf. [4, proof of Theorem 9.2]). We consider the following problem.

Problem 2.

(I) Does the assertion corresponding to Joyce–Matveev's theorem hold for surface-knots?
(II) Does the assertion corresponding to Eisermann's theorem hold for surface-knots?

Since the knot quandle does not have information of the genus of a surface-knot, we fix a non-negative integer g and consider the above problem for surface-knots of genus g. To make the problem concrete, we consider the following five conditions for a pair of surface-knots, F and F', of genus g:

(i) There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$.
(ii) There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$ such that

$$\phi_*[F] = [F'] \in H_3^{\mathrm{Q}}\big(Q(F')\big).$$

(ii′) There exists a quandle isomorphism $\phi : Q(F) \to Q(F')$ such that

$$\phi_*[F] = \pm[F'] \in H_3^{\mathrm{Q}}\big(Q(F')\big).$$

(iii) The surface-knot F is equivalent to F'.
(iii′) The surface-knot F is equivalent to F' or $-(F')^*$.

By definition, we have (iii) ⇒ (ii) ⇒ (i), (ii) ⇒ (ii′), and (iii) ⇒ (iii′). As mentioned above, we also have (iii′) ⇒ (ii′) ⇒ (i).

$$\begin{tikzcd}
\text{(i)} \arrow[dr, bend left, "\mathrm{I}_1\ ?"] & \\
\text{(ii)} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, bend right, "\mathrm{II}\ ?"'] & \text{(ii′)} \arrow[ul, Rightarrow] \arrow[d, bend left, "\mathrm{I}_2\ ?"] \\
\text{(iii)} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{(iii′)} \arrow[u, Rightarrow]
\end{tikzcd}$$

The main result of this paper is to give negative answers to Problem 2.

Theorem 3. *For a non-negative integer g, there exist arbitrarily many surface-knots of genus g such that any two of them satisfy the condition* (i) *but do not satisfy the condition* (ii′).

Theorem 4. *For a non-negative integer g, there exist infinitely many pairs of surface-knots of genus g such that each pair satisfies the condition* (ii) *but does not satisfy the condition* (iii′).

Remark 5. It follows from Theorem 3 that there exist arbitrarily many inequivalent surface-knots of genus g with the same knot group. We note that the stronger assertion is known for surface-knots of genus zero: There exist infinitely many S^2-knot with the same knot group [29].

The rest of this paper is organized as follows. In Section 2, we review the basic definitions including knot quandles and quandle cocycle invariants of surface-knots. Sections 3 and 4 are devoted to proving Theorems 3 and 4, respectively.

2. Definitions and lemmas

2.1. Quandles and knot quandles

A *quandle* [19,22], X, is a nonempty set with a binary operation $(a, b) \mapsto a * b$ satisfying the following axioms.

(Q1) For any $a \in X$, $a * a = a$.
(Q2) For any $a, b \in X$, there is a unique $c \in X$ such that $c * b = a$.
(Q3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A map $f : X \to Y$ between quandles is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$.

A *diagram* of a surface-knot is a generic projection image equipped with height information, where one of two sheets along each double point curves is broken depending on the relative height. A diagram consists of a collection of sheets, and is regarded as a compact oriented surface in $\mathbb{R}^3$. We refer to [6] for more details.

Let D be a diagram of a surface-knot F, and let $E = \{s_1, \ldots, s_m\}$ be the set of all sheets of D. Using the orientation of F and that of $\mathbb{R}^3$, we give a normal vector to each sheet. The *knot quandle* [19,22], $Q(F)$, of F is a quandle generated by $E = \{s_1, \ldots, s_m\}$ with the following defining relations. Along a double point curve, let s_j be the over-sheet and s_i (respectively, s_k) the under-sheet which is behind (respectively, in front of) the over-sheet s_j with respect to the normal vector of s_j. The defining relation is given by $s_i * s_j = s_k$ along the double point curve. We note that

$Q(F)$ is independent of the choice of the diagram of F. The following lemma will be used to construct surface-knots satisfying the condition (i).

Lemma 6. *For surface-knots F_0 and F, consider the connected sums $F_0\#F$ and $F_0\#-F^*$. Then $Q(F_0\#F)$ has the same presentation as $Q(F_0\#-F^*)$. In particular, $Q(F_0\#F)$ is isomorphic to $Q(F_0\#-F^*)$.*

Proof. A presentation of $Q(F_0\#F)$ can be obtained from that of $Q(F_0)$ and that of $Q(F)$ by adding a relation such as $a_0 = a$, where a_0 (respectively, a) is a generator of $Q(F_0)$ (respectively, $Q(F)$) corresponding to a sheet of a diagram of F_0 (respectively, F). Since $Q(F)$ has the same presentation as $Q(-F^*)$, the result follows. □

Remark 7. The above lemma does not hold for classical knots in general. Take the right-handed trefoils as k_0 and k, for example. Then it is shown in [24, p. 220] that the granny knot is not equivalent to the square knot up to orientation. (See Remark 11 for an alternative proof of this fact.) Thus we have that $Q(k_0\#k)$ is not isomorphic to $Q(k_0\#-k^*)$.

2.2. Quandle cocycle invariants

Carter et al. [3] developed the theory of quandle homology and cohomology, which was a modification of the theory of rack homology and cohomology defined by Fenn, Rourke and Sanderson [9–12]. For a quandle X and an abelian group A, we can define the chain complex $C_*^{\mathrm{Q}}(X)$ and the cochain complex $C^*_{\mathrm{Q}}(X; A)$ (of abelian groups). The nth homology group of $C_*^{\mathrm{Q}}(X)$ is called the nth *quandle homology group* [3] and is denoted by $H_n^{\mathrm{Q}}(X)$. The nth group of cocycles of $C^*_{\mathrm{Q}}(X; A)$ are denoted by $Z^n_{\mathrm{Q}}(X; A)$, and the nth cohomology group of this complex is called the nth *quandle cohomology group* [3] and is denoted by $H^n_{\mathrm{Q}}(X; A)$.

To each surface-knot F, we can associate a *fundamental class* [5] (cf. [31]), denoted by $[F]$, as an element of the third homology group $H_3^{\mathrm{Q}}(Q(F))$ of the knot quandle $Q(F)$. Since we do not use the precise construction of the fundamental class in the rest of the paper, we omit the details and introduce the definition of a quandle cocycle invariant instead. Although the quandle cocycle invariant was originally introduced as an invariant of a surface-knot, we use it as a tool for distinguishing given two fundamental classes (see Lemma 8 below).

Let F be a surface-knot and let $[F] \in H_3^{\mathrm{Q}}(Q(F))$ be the fundamental class of F. For a finite quandle X, an abelian group A and a 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(X; A)$, we define a *quandle cocycle invariant* [3], $\Phi_\theta(F)$, by

$$\Phi_\theta(F) = \sum_{c: Q(F)\to X} \big\langle c_*\big([F]\big), [\theta]\big\rangle \in \mathbb{Z}[A],$$

where $c_* : H_3^{\mathrm{Q}}(Q(F)) \to H_3^{\mathrm{Q}}(X)$ is a map induced from a quandle homomorphism $c : Q(F) \to X$, the element $[\theta]$ is a cohomology class of θ, and

$$\langle\ ,\ \rangle : H_3^{\mathrm{Q}}(X) \otimes_{\mathbb{Z}} H^3_{\mathrm{Q}}(X; A) \to A$$

is a Kronecker product. We note that the above summation is finite, since the cardinality of X is finite. The following is an easy consequence of the construction of quandle cocycle invariants, but plays an important role in the proof of Theorem 3.

Lemma 8. *For surface-knots F and F', if there exists a finite quandle X, an abelian group A and a* 3*-cocycle θ of $Z^3_{\mathrm{Q}}(X; A)$ such that*

$$\Phi_\theta(F) \neq \Phi_\theta(F') \quad \text{and} \quad \Phi_\theta(F) \neq \Phi_\theta\big(-(F')^*\big),$$

then F and F' do not satisfy the condition (ii′).

3. Proof of Theorem 3

Before proving Theorem 3, we define a pair of S^2-knots $F_{p,1}$ and $F_{p,2}$, and study their properties. For an odd prime integer p, let K_p be the 2-twist spun S^2-knot obtained from a $(2, p)$-torus knot. Let $F_{p,1}$ be the connected sum of two copies of K_p, and $F_{p,2}$ be the connected sum of K_p and $-(K_p)^*$.

For a surface-knot F and an odd prime integer p, let $\Phi_p(F)$ denote the quandle cocycle invariant of F associated with Mochizuki's 3-cocycle [23], $\theta_p \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$, of the dihedral quandle R_p and the coefficient group $\mathbb{Z}_p$. We note that the invariant $\Phi_p(F)$ takes values in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$ ($\cong \mathbb{Z}[\mathbb{Z}_p]$). As far as the author knows, there exists no general formula for the quandle cocycle invariant of the connected sum $F\#F'$ of two surface-knots F and F'. In the case of Mochizuki's 3-cocycle, however, such a formula follows from a property of the 3-cocycle θ_p mentioned in the proof of [26, Lemma 6]:

Lemma 9. $\Phi_p(F\#F') = \frac{1}{p}\Phi_p(F)\Phi_p(F') \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$.

Using Lemma 9 and Asami–Satoh's computation [1, Theorem 6.3], we have the following for an odd prime integer p:

$$\Phi_p(F_{p,1}) = p\left(\sum_{k=0}^{p-1} t^{2k^2}\right)^2 \quad \text{and} \quad \Phi_p(F_{p,2}) = p\left(\sum_{k=0}^{p-1} t^{2k^2}\right)\left(\sum_{k=0}^{p-1} t^{-2k^2}\right),$$

and also have the following for an odd prime integer p' with $p' \neq p$:

$$\Phi_p(F_{p',1}) = \Phi_p(F_{p',2}) = p.$$

Proposition 10. *If p is an odd prime integer with $p \equiv 3 \pmod 4$, then both $\Phi_p(F_{p,1})$ and $\Phi_p(-(F_{p,1})^*)$ are not equal to $\Phi_p(F_{p,2})$ in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$.*

Proof. To compare their values in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$, it is sufficient to calculate "constant terms" of them, where the constant term of $\sum_i a_i t^i$ is defined to be

$$\sum_{i \equiv 0 \pmod p} a_i \in \mathbb{Z}.$$

For integers $i, j \in \{0, \ldots, p-1\}$, it follows from the condition $p \equiv 3 \pmod 4$ that $2(i^2 + j^2) \equiv 0 \pmod p$ if and only if $(i, j) = (0, 0)$. Hence the constant term of $\Phi_p(F_{p,1})$ in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$ is equal to p. Moreover, since the invariant $\Phi_p(-(F_{p,1})^*)$ is obtained from $\Phi_p(F_{p,1})$ by replacing t with t^{-1} [4, Theorem 9.2], the constant term of $\Phi_p(-(F_{p,1})^*)$ is also equal to p.

For integers $i, j \in \{0, \ldots, p-1\}$, it is easy to see that $2(i^2 - j^2) \equiv 0 \pmod p$ if and only if $i = j$ or $i + j = p$. Hence the constant term of $\Phi_p(F_{p,2})$ in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$ is equal to $p(2p - 1)$. □

Proof of Theorem 3. We construct S^2-knots satisfying the condition of Theorem 3. Let $\mathcal{P}$ be the set of odd prime integers such that each element of $\mathcal{P}$, say p, satisfies $p \equiv 3 \pmod 4$. Since $\mathcal{P}$ is infinite, we can take n distinct primes $p_1, \ldots, p_n$ of $\mathcal{P}$ for any non-negative integer n. Given an n-tuple $I = (e_1, \ldots, e_n) \in \{1, 2\}^n$, we consider the S^2-knot

$$F_I = F_{p_1,e_1}\#\cdots\#F_{p_n,e_n},$$

and claim that these 2^n surface-knots satisfy the condition. For any two distinct elements $I = (e_1, \ldots, e_n)$ and $I' = (e'_1, \ldots, e'_n)$ of $\{1, 2\}^n$, we have $Q(F_I) \cong Q(F_{I'})$ by Lemma 6, that is, F_I and $F_{I'}$ satisfy the condition (i). Since $I \neq I'$, there exists some j ($j = 1, \ldots, n$) such that $e_j \neq e'_j$. Thus we have

$$\Phi_{p_j}(F_I) = \Phi_{p_j}(F_{p_j,e_j}) \neq \Phi_{p_j}(F_{p_j,e'_j}) = \Phi_{p_j}(F_{I'})$$

and

$$\Phi_{p_j}(F_I) = \Phi_{p_j}(F_{p_j,e_j}) \neq \Phi_{p_j}\bigl(-(F_{p_j,e'_j})^*\bigr) = \Phi_{p_j}\bigl(-(F_{I'})^*\bigr)$$

by Lemma 9 and Proposition 10. Hence F_I and $F_{I'}$ do not satisfy the condition (ii′) by Lemma 8.

When the genus g is greater than zero, we consider the connected sum of F_I and a trivial surface-knot of genus g. Then these 2^n surface-knots of genus g satisfy the condition of Theorem 3. □

Remark 11. We give an alternative proof of the fact mentioned in Remark 7. By the above proof, $F_{3,1}$ $(= K_3\#K_3)$ is not equivalent to $F_{3,2}$ $(= K_3\# -(K_3)^*)$. Then, for the right-handed trefoil knot (i.e., (2, 3)-torus knot) k_3, it follows from [21] that $k_3\#k_3$ is not equivalent to $k_3\# -(k_3)^*$. Since the trefoil knot is invertible, the granny knot, $k_3\#k_3$, is not equivalent to the square knot, $k_3\#(k_3)^*$, up to orientation.

4. Proof of Theorem 4

The proof is divided into two cases: One is the case where $g = 0$ and the other is the case where $g > 0$.

4.1. $g = 0$ case

Take integers $n, p, q > 5$ such that p and q are relatively prime. Let K be an n-twist spun S^2-knot obtained from a (p, q)-torus knot, and $\widehat{K}$ be an S^2-knot obtained from K by Gluck surgery [13]. We remark that the exterior $E(K)$ of the S^2-knot K is homeomorphic to the exterior $E(\widehat{K})$ of $\widehat{K}$. It is shown in [14] that the ambient space of $\widehat{K}$ is homeomorphic to the 4-sphere S^4 and that $\widehat{K}$ is not equivalent to K up to orientation. In particular, K and $\widehat{K}$ do not satisfy the condition (iii′).

Let Σ be the trivial surface-knot of genus two, and consider the pair of surface-knots $K\#\Sigma$ and $\widehat{K}\#\Sigma$. We notice that the exterior $E(K\#\Sigma)$ is homeomorphic to $E(\widehat{K}\#\Sigma)$. Then $\widehat{K}\#\Sigma$ is equivalent to $K\#\Sigma$, since a surface-knot of genus greater than one is determined by its exterior [16]. Hence we have

$$Q(K) > \xrightarrow[\cong]{\phi_1} Q(K\#\Sigma) \xrightarrow[\cong]{\phi_2} Q(\widehat{K}\#\Sigma) \xrightarrow[\cong]{\phi_3} Q(\widehat{K})$$

and

$$(\phi_3 \circ \phi_2 \circ \phi_1)_*[K] = (\phi_3 \circ \phi_2)_*[K\#\Sigma] = (\phi_3)_*[\widehat{K}\#\Sigma] = [\widehat{K}],$$

where the map ϕ_1 (respectively, ϕ_3) is induced by doing the connected sum of the trivial surface-knot Σ to K (respectively, $\widehat{K}$), and the map ϕ_2 is induced from the equivalence between $K\#\Sigma$ and $\widehat{K}\#\Sigma$. When we vary integers n, p and q, we can obtain infinitely many such pairs.

4.2. $g > 0$ case

Let $T(k)$ denote the spun T^2-knot obtained from a nontrivial classical knot k, and let $\tilde{T}(k)$ denote the turned spun T^2-knot obtained from k. Take a ribbon surface-knot G of genus $g - 1$ $(\geqslant 0)$ and consider the pair of surface-knots $G\#T(k)$ and $G\#\tilde{T}(k)$ of genus g. It is easy to see that these two surface-knots satisfy the condition (ii). We note that the fundamental classes of them are equal to zero elements.

To distinguish them, we use Kawauchi's Gauss sum invariant [20, p. 1047], $\varsigma(F) \in \mathbb{Z}$, of a surface-knot F. It is shown in [20] that $\varsigma(G) = 2^{g-1}$, $\varsigma(T(k)) = 2$ and $\varsigma(\tilde{T}(k)) = 0$. Using the connected sum formula [20, Theorem 1.2]

$$\varsigma(F_1\#F_2) = \varsigma(F_1)\varsigma(F_2),$$

we have

$$\varsigma\big(G\#T(k)\big) = 2^g \neq 0 = \varsigma\big(G\#\tilde{T}(k)\big),$$

and it follows that they do not satisfy the condition (iii′). When we vary a nontrivial classical knot k, we can obtain infinitely many such pairs.

Remark 12. We may take any surface-knot G of genus $g - 1$ as long as it satisfies the condition $\varsigma(G) \neq 0$, though we take a ribbon surface-knot as G in the above proof.

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